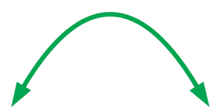
MATH1520 University Mathematics for Applications **Fall 2021**

Chapter 7: Application of Derivatives II

Learning Objectives:
(1) Discuss concavity.
(2) Use the sign of the second derivative to find intervals of concavity.
(3) Locate and examine inflection points.
(4) Apply the second derivative test for relative extrema.
(5) Determine horizontal and vertical asymptotes of a graph.
(6) Discuss and apply a general procedure for sketching graphs.

7.1 Concavity and points of inflection

Intuitively: On the $x - y$ plane: when a curve, or part of a curve, has the shape:



concave

we say that the shape is concave downward. On the other hand, if it takes the shape

convex

we say that it is concave upward.

Remark. In some textbooks "concave upward" is called ~~concave~~ up or convex; "concave downward" is called ~~concave~~ down or concave.
convex
convex

Definition 7.1.1. If the function $f(x)$ is differentiable on the interval (a, b), then *the graph* of f is

(strictly convex)
(i) strictly concave upward on (a, b) if $f'(x)$ is strictly increasing on the interval. In particular, if f is second-differentiable, the condition is equivalent to $f''(x) > 0$.

(strictly concave)
(ii) strictly concave downward on (a, b) if $f'(x)$ is strictly decreasing on the interval. In particular, if f is second-differentiable, the condition is equivalent to $f''(x) < 0$.

(convex)
(iii) concave upward on (a, b) if $f'(x)$ is increasing on the interval. In particular, if f is second-differentiable, the condition is equivalent to $f''(x) \geq 0$.

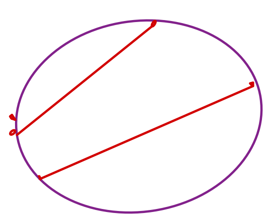
(concave)
(iv) concave downward on (a, b) if $f'(x)$ is decreasing on the interval. In particular, if f is second-differentiable, the condition is equivalent to $f''(x) \leq 0$.

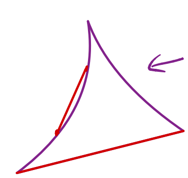
In case (i)/(iii), the function f is said to be *strictly convex/convex*; in case (ii)/(iv), f is said to be *strictly concave/concave*.

Remark. 1. In some calculus texts, what we called "strictly convex/concave" above is called "convex/concave", and what we called " convex/concave" above is called "weakly convex/concave"

2. General definition of convexity/concavity of continuous curves on a plane via secant lines:

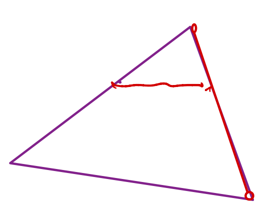
• *For a closed curve $C \subset \mathbb{R}^2$:* C is strictly convex if all secant lines to C lies in the "inside" except for the end points. e.g. A circle is strictly convex.





← concave: secant lines lie away from the "inside" of C

E.g. A piecewise convex curve:

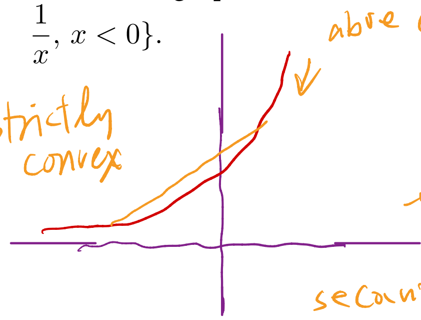


convex: all secant lines lie out of the "outside region" of C

• *For the graph C of a continuous function f on the $x-y$ plane:* f is concave if all secant lines to the graph do not intercept the "upside component" of $\mathbb{R}^2 \backslash C$. E.g. $C = \{(x, y) \,|\, f(x) = \frac{1}{x}, x < 0\}$.

above C

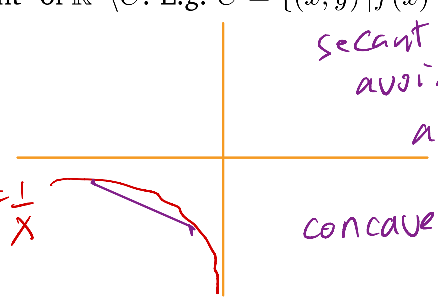
strictly convex



y = f(x)

on C.

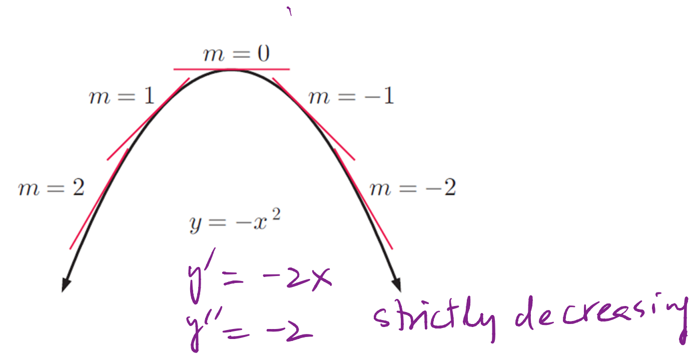
secant lines lie above C intersect only at endpts



secant lines avoid the region above C

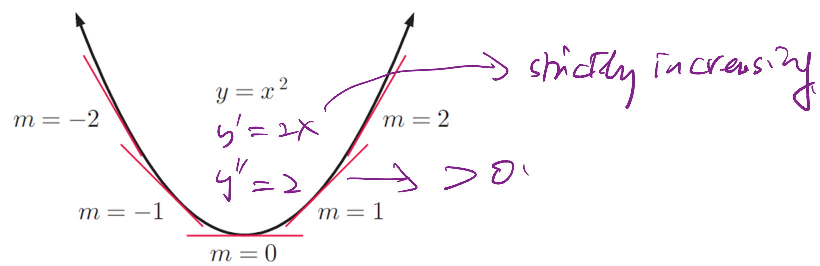
concave graph

A test for shapes of graphs:



As x increases, $f'(x)$ is $\downarrow$

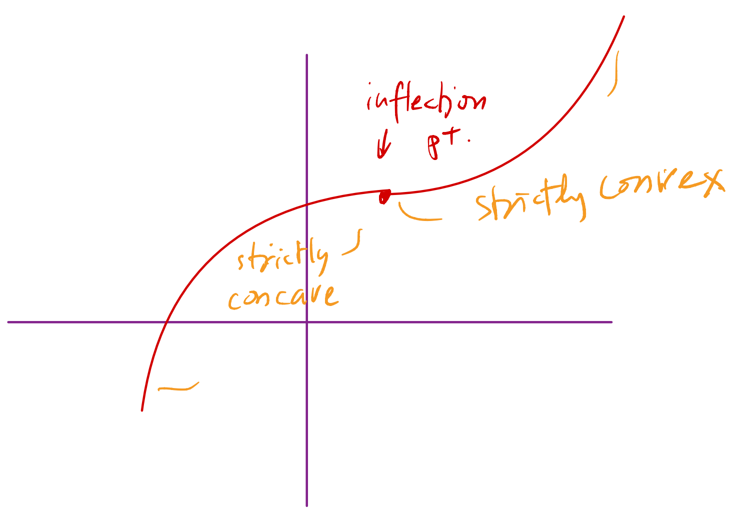
$f''(x) = -2 < 0$ for strictly concave downward curve.



As x increases, $f'(x)$ is $\uparrow$

$f''(x) = 2 > 0$ for strictly concave upward curve.

Definition 7.1.2. If $f(x)$ changes strict concavity at some point c in the domain, then the point $(c, f(c))$ on the $x - y$ plane is called an *inflection point* of the graph of f.



Procedure for Determining Intervals of Concavity & Inflection Points:

Suppose the function $f(x)$ is such that f'' is piecewise continuous.

1. Find all c for which $f''(c) = 0$ or $f''(c)$ does not exist, and divides the domain into several intervals.
2. For each interval,

   strictly

   - if $f''(x) > 0$, the graph of $f(x)$ is strictly concave upward. (I.e. f is a convex function.)
   - if $f''(x) < 0$, the graph of $f(x)$ is strictly concave downward. (I.e. f is a concave function.)

   strictly
3. For all c found in step 1,
   - if $f''(x)$ changes sign on two sides of c, then $(c, f(c))$ is an inflection point on the graph of f;
   - otherwise, $(c, f(c))$ is not an inflection point on the graph of f.

Example 7.1.1.

$$f(x) = x^3 + 1 \qquad f' = 3x^2$$

$$f''(x) = 6x = 0 \quad \Rightarrow \quad x = 0.$$

- if $x < 0$, $f''(x) < 0$, $\Rightarrow$ f is strictly concave on $(-\infty, 0)$;
- if $x > 0$, $f''(x) > 0$, $\Rightarrow$ f is strictly convex on $(0, \infty)$.

Since $f''(x)$ changes signs on both sides of $x = 0$, $(0, 1)$ is the unique inflection point on the graph of f.

$x=0,\ y=f(0)=1$

Example 7.1.2. Describe the concavity and find all inflection points of the graph of $f(x) = 2x^6 - 5x^4 + 7x - 3$.

		-1		0		1	
$x+1$	$-$		$+$		$+$		$+$
$x-1$	$-$		$-$		$-$		$+$
f''	$+$		$-$		$-$		$+$

Solution. $f' = 12x^5 - 20x^3 + 7$

$$f''(x) = 60x^4 - 60x^2 = 60x^2(x^2 - 1) = \underbrace{60x^2}_{>0 \text{ when } x \neq 0}(x-1)(x+1) = 0 \quad \Rightarrow \quad x = 0, \pm 1.$$

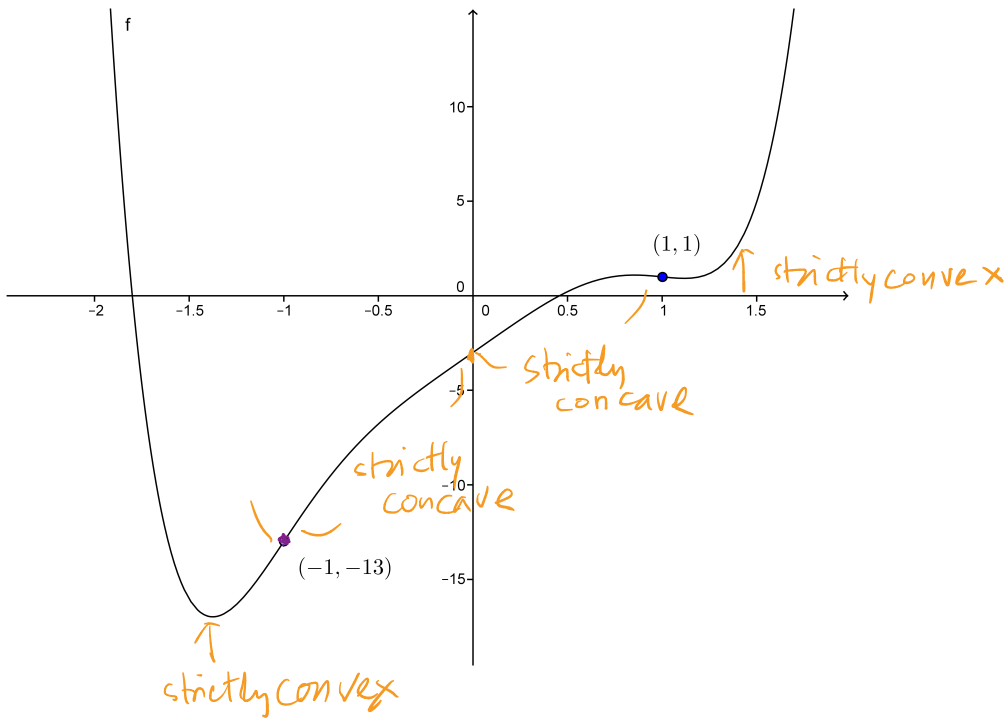
x	$(-\infty, 0)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, +\infty)$
$f''(x)$	$+$	0	$-$	0	$-$	0	$+$
concavity	up($\smile$)		down($\frown$)		down($\frown$)		up($\smile$)
	strictly convex		strictly concave		strictly concave		strictly convex

Two inflection points: $(-1, -13)$, $(1, 1)$.
($(0, -3)$ is not an inflection point!)

$f(-1) = 2 - 5 - 7 - 3 = -13$
$f(1) = 2 - 5 + 7 - 3 = 1$

inflection pt. occurs

two inflection points: $(-1, f(-1)) = (-1, -13)$ $\quad (1, f(1)) = (1, 1)$



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Remark.

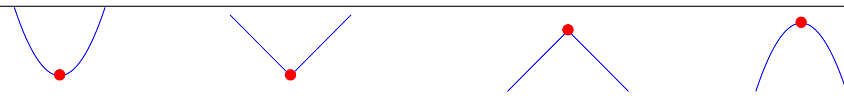
- c is a critical point $\iff$ $f'(c) = 0$ or $f'(c)$ does not exist
  c is a critical point $\left\{\begin{matrix}\Leftarrow \\ \not\Rightarrow\end{matrix}\right\}$ f' changes sign at c
- $(c, f(c))$ is an inflection point on the graph of f $\iff$ f'' changes sign at c
  $(c, f(c))$ is an inflection point on the graph of f $\left\{\begin{matrix}\Rightarrow \\ \not\Leftarrow\end{matrix}\right\}$ $f''(c) = 0$ or undefined

Theorem 7.1.1 (The Second Derivative Test: Relative Extrema).

Suppose $f'(a) = 0$!

1. *If $f''(a) < 0$, then f has a relative maximum at a.*
2. *If $f''(a) > 0$, then f has a relative minimum at a.*
3. *If $f''(x) = 0$, we have no conclusion.*

previously when $f'(a)=0$
a might be a local extremum
a is local max when the sign of f' changes from + to −
" local min " − to +
"



	relative min	relative min	relative max	relative max
	$f'(a) = 0$	$f'(a)$ does not exist	$f'(a)$ does not exist	$f'(a) = 0$
1st test:	$-\quad +$	$-\quad +$	$+\quad -$	$+\quad -$
2nd test:	$f''(a) > 0$	Not Applicable	Not Applicable	$f''(a) < 0$

Example 7.1.3.

$$f(x) = \frac{1}{30}x^6 - \frac{1}{12}x^4.$$

$x-\sqrt{5/3}$ $-$ $-$ $-$ $+$

$x+\sqrt{5/3}$ $-$ $+$ $+$ $+$

Use the first and second derivative test to study the relative extrema.

x^3 $-$ $-$ $+$ $+$

f' $-$ $-\sqrt{5/3}$ $+$ 0 $-$ $\sqrt{5/3}$ $+$

Solution.

$= \frac{1}{5}x^3\left(x^2 - \frac{5}{3}\right)$

$$f'(x) = \frac{1}{5}x^5 - \frac{1}{3}x^3 = \frac{1}{5}x^3\left(x+\sqrt{\frac{5}{3}}\right)\left(x-\sqrt{\frac{5}{3}}\right) = 0 \quad \Rightarrow \quad x = -\sqrt{\frac{5}{3}}, 0, \sqrt{\frac{5}{3}}$$

$$f''(x) = x^2(x+1)(x-1).$$

$= x^4 - x^2 = x^2(x^2-1)$

x	$(-\infty, -\sqrt{\frac{5}{3}})$	$-\sqrt{\frac{5}{3}}$	$(-\sqrt{\frac{5}{3}}, 0)$	0	$(0, \sqrt{\frac{5}{3}})$	$\sqrt{\frac{5}{3}}$	$(\sqrt{\frac{5}{3}}, +\infty)$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f''(x)$		$f'' > 0$ $\frac{5}{3}\left(\frac{5}{3}-1\right)$		$f'' = 0$		$f'' > 0$ $\frac{5}{3}\left(\frac{5}{3}-1\right)$	
1st test:		relative min		relative max		relative min	
2nd test:		relative min		inconclusive		relative min	

local max local min.

$x-1$ $-$ $-$ $+$

$x+1$ $-$ $+$ $+$

f' $+$ -1 $-$ 1 $+$

$f''<0$ $f''>0$

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Exercise 7.1.1. Apply the first and the second derivative tests to find the local maxima/minima and the global maximum/minimum of $f(x) = x^3 - 3x$.

$f' = 3x^2 - 3 = 3(x+1)(x-1)$

$= 0$ when $x = \pm 1$ $\leftarrow$ critical numbers (potential local extrema)

$f'' = 6x$: local max at $x = -1$.

local min at $x = 1$

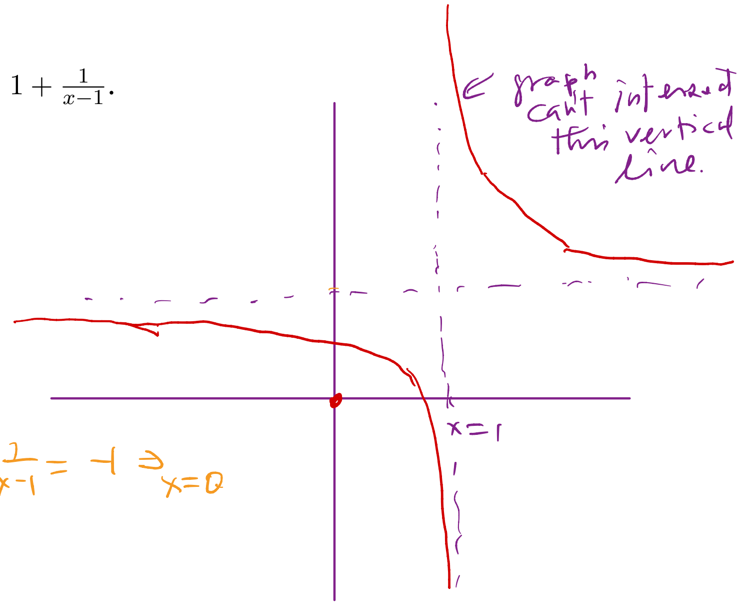
no abs max

no abs min.

$f(-1), f(1)$ finite. $\lim_{x\to+\infty} f(x) = +\infty$ $\lim_{x\to-\infty} f(x) = -\infty$

7.2 Curve sketching

Example 7.2.1. Sketch the graph of $y = f(x) = 1 + \frac{1}{x-1}$.



Solution.

Step 1. Analyze $f(x)$.

1. domain: $\{x \in \mathbb{R} \,|\, x \neq 1\}$

2. x, y intercepts:
   Let $x = 0$, then $y = 0$;
   Let $y = 0$, then $x = 0$. $1 + \frac{1}{x-1} = 0 \quad \frac{1}{x-1} = -1 \Rightarrow x = 0$
   $\Rightarrow$ only one intercept: $(0, 0)$

3. vertical and horizontal asymptotes:

$$\lim_{x\to 1^+} f(x) = +\infty, \lim_{x\to 1^-} f(x) = -\infty \quad \Rightarrow \quad \text{vertical asymptote: } x = 1$$
$$\lim_{x\to +\infty} f(x) = 1, \lim_{x\to -\infty} f(x) = 1 \quad \Rightarrow \quad \text{horizontal asymptote: } y = 1.$$

Step 2. Analyze $f'(x)$.

$$f'(x) = -\frac{1}{(x-1)^2}, x \neq 1.$$

1. interval where f is strictly increasing: none $\quad (f'(x) < 0$ in the domain$)$
   interval where f is strictly decreasing: $(-\infty, 1)$, $(1, +\infty)$

2. critical points of f: none $\quad (x = 1$ is not in the domain$)$

3. relative extrema of f: none

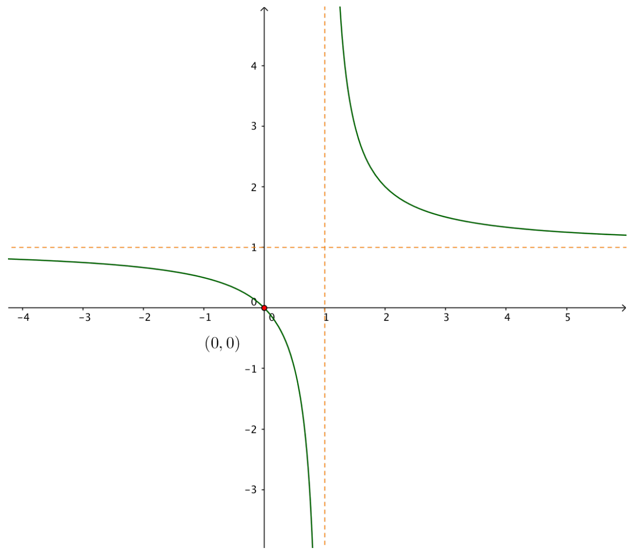
Step 3. Analyze $f''(x)$.

$$f''(x) = \frac{2}{(x-1)^3}, x \neq 1.$$

1. interval where f is strictly convex: $(1, +\infty) \quad (f'' > 0)$
   interval where f is strictly concave: $(-\infty, 1) \quad (f'' < 0)$

2. inflection points on the graph: none $\quad (x = 1$ is not in the domain$)$

Step 4. Sketch.



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Definition 7.2.1 (Asymptotes)**.**

the line $x = c$ is a vertical asymptote of the graph of $f(x)$

$$\text{if } \lim_{x \to c^-} f(x) \text{ or } \lim_{x \to c^+} f(x) \text{ is } +\infty \text{ or } -\infty;$$

the line $y = b$ is called a horizontal asymptote of the graph of $f(x)$

$$\text{if } \lim_{x \to -\infty} f(x) \text{ or } \lim_{x \to +\infty} f(x) \text{ is } b.$$

Note: It may happen that both $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ exist, but they are not the same.

A General Procedure for Sketching the Graph of $f(x)$

Step 1. Analyze $f(x)$:
(1) domain, (2) x, y intercepts, (3) vertical / horizontal asymptotes of the graph.

Step 2. Analyze $f'(x)$:
(1) intervals where f is increasing / decreasing, (2) critical points of f (3) relative extrema of f

Step 3. Analyze $f''(x)$:
(1) intervals of where f is convex/concave, (2) inflection points on the graph

Step 4. Sketch:

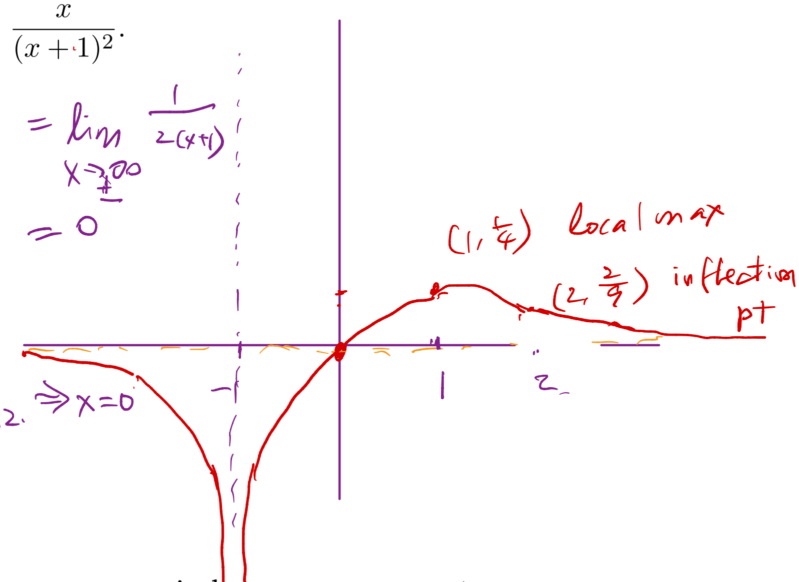
First label all asymptotes, intercepts, critical points, inflection points, then sketch the graph.

Example 7.2.2. Sketch the graph of

$$f(x) = \frac{x}{(x+1)^2}.$$

$\lim_{x\to+\infty} f = \lim_{x\to\pm\infty} \frac{1}{2(x+1)}$

$= 0$



Solution.

Step 1. Analyze $f(x)$.

1. domain: $\{x \in \mathbb{R} \,|\, x \neq -1\}$
2. x, y intercepts:
   Let $x = 0$, then $y = 0$;
   Let $y = 0$, then $x = 0$.
   $\Rightarrow$ only one intercept: $(0, 0)$

   $0 = \frac{x}{(x+1)^2} \Rightarrow x = 0$
3. vertical and horizontal asymptotes:

$$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^-} f(x) = -\infty \quad\Rightarrow\quad \text{vertical asymptote: } x = -1$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0 \quad\Rightarrow\quad \text{horizontal asymptote: } y = 0.$$

$= \frac{1}{(x+1)^2} + x\left(-2\frac{1}{(x+1)^3}\right) = \frac{x+1-2x}{(x+1)^3}$

Step 2. Analyze $f'(x)$.

$$f'(x) = \frac{1-x}{(x+1)^3} = 0 \quad\Rightarrow\quad x = 1.$$

$(x+1)^3$	$-$	$+$		$+$
$1-x$	$+$	$+$		$-$

x	$(-\infty, -1)$	$(-1, 1)$	1	$(1, +\infty)$
$f'(x)$	$-$	$+$	0	$-$
$f(x)$	$\downarrow$	$\uparrow$	max: 1	$\downarrow$

$f(1) = \frac{1}{4}$

only one critical point: 1 (with corresponding critical value $\frac{1}{4}$), at which a relative maximum occurs. ($x = -1$ is not in the domain.)

$(-1)\frac{1}{(x+1)^3} + (1-x)\left(-3\frac{1}{(x+1)^4}\right)$

Step 3. Analyze $f''(x)$.

$$f''(x) = \frac{2(x-2)}{(x+1)^4} = 0 \quad\Rightarrow\quad x = 2.$$

$f(2) = \frac{2}{9}$

$\vee$ 0 when f'' is defined

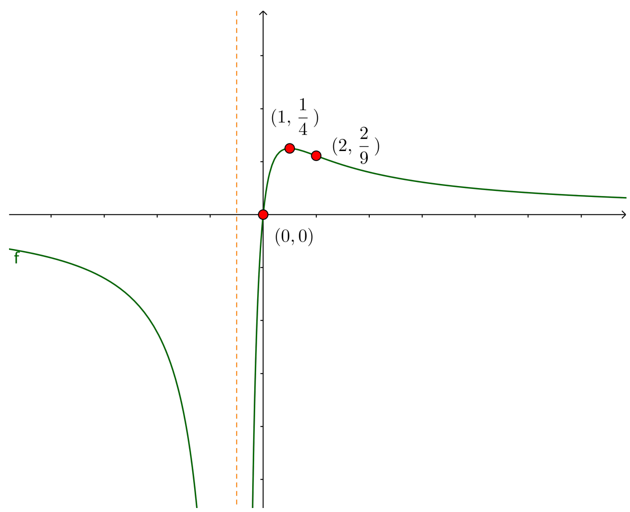
$f'' > 0$ when $x > 2$,

< 0 when $x < 1$

x	$(-\infty,-1)$	$(-1,2)$	2	$(2,+\infty)$
$f''(x)$	$-$	$-$	0	$+$
graph of $f(x)$	$\frown$	$\frown$	inflection point	$\smile$

inflection point: $(2, \frac{2}{9})$

Step 4. Sketch.



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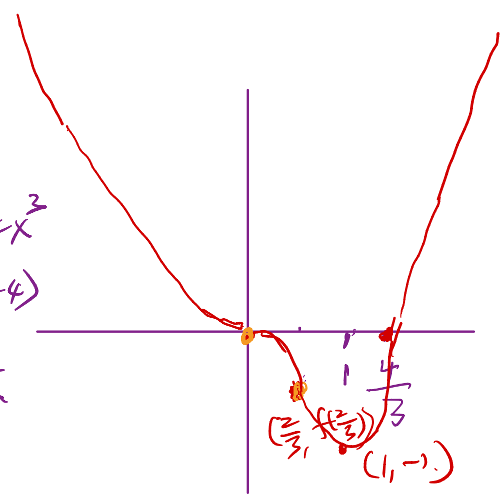
Exercise 7.2.1. Sketch the graph of $3x^4 - 4x^3$. $= f(x)$



Domain: $\mathbb{R}$

graph intersects the y-axis when $x=0$ $f(0)=0$

intersects the x-axis when $y=0$ $0 = 3x^4-4x^3$

$= x^3(3x-4)$

$\Rightarrow x = 0, \frac{4}{3}$

$f = x^3(3x-4)$

$f(\frac{2}{3}) = (\frac{2}{3})^3(2-4) < 0$

$\lim_{x\to\pm\infty} f = \lim_{x\to+\infty}(3x^4-4x^3) = +\infty$

$f' = 12x^3 - 12x^2 = 12x^2(x-1) = 0 \Rightarrow x = 0, 1.$ crit pts

f': $-$ (0) $-$ (1) $+$

local min

$f(1) = 3-4 = -1$

f convex concave convex

$f'' = 12(3x^2-2x) = 12x(3x-2) = 0$

when $x=0$ or $x = \frac{2}{3}$

inflection pts at $(0, f(0))$, $(\frac{2}{3}, f(\frac{2}{3}))$

$3x-2$	$-$		$-$	$\frac{2}{3}$	$+$
x	$-$	0	$+$		$+$
f''	$+$		$-$		$+$